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LETTER TO THE EDITOR

Integrability of interacting two-level boson systems

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Abstract

We prove that a wide class of two-level boson systems interacting through k-body forces is integrable in the semiclassical limit irrespective of the value of the parameters of the model. This class of models includes number-conserving Hamiltonians, and interactions which do not conserve the particle number. Applications to two-component and atom–molecule Bose–Einstein condensates at zero temperature as well as to the bosonic k-body embedded ensembles of Gaussian random matrices in the dense limit are discussed.

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Exactly solvable models play an important role in all branches of physics. In some instances, they provide the basic understanding upon which we build up intuition. This is the philosophy behind mean-field theories. In others, they are an effective tool to test the approximate methods used in more elaborated descriptions of the physical situation. In this case, having at hand an exact solution permits us to understand the convergence properties of the approximations involved. To quote some examples, we mention the two-level Lipkin model [1], the paring interaction [2] and the Calogero–Sutherland Hamiltonian [3]. Though the connection is not so clear, many—if not all—of the exactly solvable models are integrable. That is, there are as many independent integrals of motion as degrees of freedom in the system. Indeed, new exact and integrable pairing-like models have been recently discovered [4–6]. For obvious reasons any non-trivial model displaying integrability is of broad interest, in particular if the model has the least number of restrictions (maximum freedom) on the specification of the parameters that define it. In this letter, we prove integrability of a large class of interacting two-level boson systems, with no assumptions on the parameters defining the interaction (except for the Hermiticity of the Hamiltonian) and weak restrictions on the specific type of interaction. We show that these types of models, in the semiclassical limit, correspond to systems of two degrees of freedom.

Consider n spinless bosons distributed on $l = 2$ single-particle states interacting through a potential that couples up to k bodies ($k \leqslant n$). We shall refer to k as the rank of the interaction. For simplicity in the presentation we shall consider the case $k = 2$ first, which is the most

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interesting one from the physical point of view, and later generalize the results. The *most general* Hamiltonian describing this situation, expressed in the number representation on a second quantized form, is given by

$$\hat{H} = \sum_{\alpha} \epsilon_{\alpha} b_{\alpha}^{\dagger} b_{\alpha} + \sum_{\substack{\alpha \leqslant \beta \\ \gamma \leqslant \delta}} v_{\alpha,\beta;\gamma,\delta} b_{\alpha}^{\dagger} b_{\beta}^{\dagger} b_{\gamma} b_{\delta}. \tag{1}$$

Here, $b_{\alpha}^{\dagger}$ creates a boson in the single-particle level α ($\alpha = 1, 2$) and b_{α} destroys it. These operators obey the usual bosonic commutation rules. The single-particle energies are denoted by ϵ_{α} and the (Hermitian) two-body matrix elements are $v_{\alpha,\beta;\gamma,\delta}$. Considering for simplicity only real coefficients, the Hamiltonian of equation (1) has eight parameters that defines it. From these, the two single-particle energies ϵ_{α} define the energy range and scale; therefore, apart from the scale and the zero of the energy, $\hat{H}$ is defined through six independent (real) parameters. Clearly, the Hamiltonian $\hat{H}$ conserves the number of particles n in the system, i.e., $\hat{H}$ commutes with the particle number operator $\hat{n} = \sum_{\alpha} b_{\alpha}^{\dagger} b_{\alpha}$. The Hilbert space dimension N is given by $N = n+1$. Hamiltonians of the type of equation (1) are currently under investigation in the context of two-component Bose–Einstein condensates (BEC) at zero temperature [7, 8].

One possibility of obtaining information on the dynamical behaviour of Hamiltonians of the form of $\hat{H}$ is to consider their semiclassical limit. Being bosons, the elementary single-particle excitations can be assumed of the type of the harmonic oscillator. In atomic and molecular spectroscopy, algebraic Hamiltonians of the type $\hat{H}$ are typically written in a symmetric form with respect to the ordering of the creation and annihilation operators. This permits the zero-point energy to be defined consistently, which in turn allows for an energy-to-energy comparison of the classical and quantum systems in a systematic way. Equation (1) is manifestly non-symmetric with respect to the ordering, but it can be brought to a symmetric form by exploiting the commutation relations of the bosonic creation and annihilation operators, typically in the form $b_{\alpha}^{\dagger} b_{\beta} = \left(b_{\alpha}^{\dagger} b_{\beta} + b_{\beta} b_{\alpha}^{\dagger} - \delta_{\alpha,\beta}\right)/2$. Following [9, 10], once the quantum Hamiltonian $\hat{H}$ is expressed in a symmetric form, the corresponding classical Hamiltonian is obtained using Heisenberg correspondence relations [11]

$$b_{\alpha}^{\dagger} \longrightarrow I_{\alpha}^{1/2} \exp(\mathrm{i}\phi_{\alpha}) \qquad b_{\alpha} \longrightarrow I_{\alpha}^{1/2} \exp(-\mathrm{i}\phi_{\alpha}) \tag{2}$$

where ϕ_{α} is an angle and I_{α} is its canonically associated momentum. These variables correspond to the action-angle variables of the harmonic oscillator, and not to the action-angle variables of $\hat{H}$. Using Heisenberg relations yields a classical Hamiltonian with a zero-point energy correction, which we denote below as E_0. This constant term permits a direct comparison between the classical and quantum energies, and corresponds to having all elementary (harmonic) excitations set to $1/2$.

Using equations (2) in the symmetrized form of the Hamiltonian $\hat{H}$, the classical Hamiltonian H attains the generic form

$$H(I_1, I_2, \phi_1, \phi_2) = H_0(I_1, I_2) + V(I_1, I_2, \phi_1, \phi_2) - E_0. \tag{3}$$

The interaction has been written separating an angle-independent part H_0, which is obviously fully integrable, the residual interaction V which also depends upon the angle variables, and the zero-point energy E_0. These terms are explicitly given by

$$H_0 = \omega_1 I_1 + \omega_2 I_2 + v_{11,11} I_1^2 + v_{12,12} I_1 I_2 + v_{22,22} I_2^2 \tag{4}$$

$$V = 2[\omega_{12} + v_{11,12} I_1 + v_{12,22} I_2](I_1 I_2)^{1/2} \cos(\phi_1 - \phi_2) + 2 v_{11,22} I_1 I_2 \cos 2(\phi_1 - \phi_2) \tag{5}$$

$$E_0 = (2\epsilon_1 + 2\epsilon_2 - 3v_{11,11} - v_{12,12} - 3v_{22,22})/4. \tag{6}$$

In equations (4) and (5), we have introduced $\omega_1 = \epsilon_1 - 2v_{11,11} - v_{12,12}/2$, $\omega_2 = \epsilon_2 - 2v_{22,22} - v_{12,12}/2$ and $\omega_{12} = -(v_{11,12} + v_{12,22})$.

Some important points of this classical system are worth noting. First, the classical–quantum comparison is semiclassical because Schrödinger quantization ($\phi \to \phi, I \to -i\hbar\partial/\partial\phi$) of general coordinates is semiclassical, and the order of the harmonic oscillator action and angle variables is not specified. We refer the reader to [9] where this issue is extensively discussed. Secondly, the number of degrees of freedom of the classical system corresponds to the number of single-particle levels l, which in the present case is two. Indeed, by considering the second quantized form of the Hamiltonian, emphasis is put on the number of field excitations, which are of harmonic oscillator type. As usual in these cases, the semiclassical limit is attained in the limit $N \to \infty$, which in this case is given in terms of the number of particles n.

From equation (5) we observe that V depends only upon the angle variables through the combination $\phi_1 - \phi_2$, and consequently, the same angle dependence appears in the classical equations of motion. This angle relation corresponds to a 1:1 resonance. Clearly, this term is invariant under translations of the angle variables, and associated with this symmetry there is a constant of motion. We exploit this by performing a canonical transformation to new angles χ and ψ, and their canonically conjugated momenta K and J respectively. The canonical transformation is defined by the generating function $W = K\phi_1 + J(\phi_2 - \phi_1)$. Hence

$$I_1 = K - J \qquad I_2 = J \tag{7a}$$

$$\chi = \phi_1 \qquad \psi = \phi_2 - \phi_1. \tag{7b}$$

From equations (7) it follows that the new Hamiltonian H' depends only on the new angle variable $\psi = \phi_2 - \phi_1$. Therefore, χ is a cyclic coordinate in H' and consequently its canonically-conjugated momentum K is a constant of motion. Equations (7a) imply that $K = I_1 + I_2$. Constants of motion of this type correspond to the polyads of algebraic Hamiltonians in molecular physics [9, 10], and they appear in more general contexts than the 1:1 resonance of the present case. We shall come to this point later.

To have a physical insight into K, and to show that this constant of motion is actually in involution with the classical Hamiltonian, consider the number operator $\hat{n}$. Writing it in a symmetrized form $\hat{n} = \sum_\alpha \left(b_\alpha^\dagger b_\alpha + b_\alpha b_\alpha^\dagger - 1\right)/2 = n$. Using Heisenberg correspondence relations, equations (2), it is straightforward to obtain $n = I_1 + I_2 - 1 = K - 1$. Therefore, the polyad K corresponds to $n + 1$, which is identical to N, the dimension of Hilbert space in the quantum case. Since n can be arbitrarily fixed (independently of the excitation energy), it is obviously in involution with the classical Hamiltonian. This can also be checked by verifying that the Poisson bracket $\{K, H\}$ vanishes. Since we have two independent integrals of motion in involution, H' and K, and the system has two degrees of freedom, we have proved that the Hamiltonian H is integrable in the sense of Liouville [12]. We emphasize that no assumptions have been made on the specific values of the parameters ϵ_α and $v_{\alpha,\beta;\gamma,\delta}$. This implies that the property of integrability holds for *all choices*.

One may ask whether the system investigated so far is a special case or is it significantly different from those already known: the Lipkin model, the pairing interaction, or the new Hamiltonians which have been recently found to be exactly solvable [5, 6]. To clarify this, we emphasize the general character of the (spinless) two-body interaction of equation (1). This is to be contrasted with the pairing type of interaction (seniority conserving) of the above-mentioned models. This is clear from the absolute freedom to specify the parameters defining the interaction, while the other models only depend on one parameter. Yet, there is a formal analogy with the Lipkin model (degenerate fermions distributed on two degenerate levels) and our two-level interacting boson system. As we shall proceed to show, we can actually allow

modifications on the type of interaction, and prove integrability in systems interacting through k-body forces, and in systems that do not conserve the number of particles.

Consider first the general k-body interaction on a two-level boson system that conserves the number of particles. The contribution of rank r is a linear combination (on the labels s and t below) of terms proportional to

$$\left(b_1^\dagger\right)^s \left(b_2^\dagger\right)^{r-s} (b_1)^t (b_2)^{r-t}. \tag{8}$$

Here, s denotes the number of bosons that are created in the first level and t the number of bosons that are annihilated from it ($r - s$ are created and $r - t$ are destroyed in the second one). Note that the number of parameters that define the model increases quadratically with k. Proceeding as before, namely symmetrizing the k-body Hamiltonian and using Heisenberg correspondence relations, it is straightforward to show that the dependence upon the classical angles ϕ_1 and ϕ_2 will be *only* through terms of the form $\cos[(s - t)(\phi_1 - \phi_2)]$. These terms display the same invariance with respect to translations of the angles, and therefore $K = I_1 + I_2$ will also be an independent constant of motion.

The above generalization to the general k-body interaction exploits the occurrence of the 1:1 resonance mentioned above, that is, that the angles ϕ_1 and ϕ_2 appear in the Hamiltonian only as a function of $\phi_1 - \phi_2$. More general situations can be considered, and integrability can be proved in the same way if the interaction terms are such that they define at most one resonance. Such interaction terms appear, e.g., in algebraic Hamiltonians in molecular physics and in the context of laser-induced excitations. We therefore consider Hamiltonians constructed as a linear combination (on the labels s and t) of terms $h_{st}^{(k)}$ given by

$$\hat{h}_{st}^{(k)} = v_{s,t} \left(b_1^\dagger\right)^{ps} \left(b_2^\dagger\right)^{q(k-s)} (b_1)^{pt} (b_2)^{q(k-t)} \tag{9}$$

with p and q non-equal, and $s, t = 0, \ldots, k$. The parameters $v_{s,t}$ are only restricted to guarantee Hermiticity of the Hamiltonian ($v_{t,s}^* = v_{s,t}$). Obviously, this type of interaction does not commute with the particle number operator $\hat{n}$ (unless $p = q$), and consequently the number of particles is not conserved. For this interaction the angle variables appear in the classical Hamiltonian only as a function of $p\phi_1 - q\phi_2$. Ignoring zero-point corrections which are not relevant now, the classical conserved quantity is $K = qI_1 + pI_2$, and its quantum counterpart corresponds to the operator $\hat{K} = qb_1^\dagger b_1 + pb_2^\dagger b_2$. It can be readily checked that the commutators $\left[\hat{K}, \hat{h}_{st}^{(k)}\right]$ and the Poisson brackets $\left\{K, h_{st}^{(k)}\right\}$ vanish.

These results, in particular for the particle-conserving k-body interaction, can be understood from an algebraic point of view. Indeed, a convenient quantum-mechanical treatment of the k-body interacting two-level boson model exploits the $SU(2)$ algebra structure of the problem, in analogy with the Lipkin model [13]. The Hamiltonian is written as a polynomial of the angular momentum operators whose degree depends on k, and is worked out in the usual basis $|j, m\rangle$. This crucially depends on the conservation of the number of particles: the square of the total angular momentum is related to the number of bosons by $J^2 = j(j+1)$, with $j = n/2$. Therefore, the classical motion takes place in the space of angular momentum on the sphere of radius $j(j + 1)$, where the precise trajectory is determined by intersecting the surface induced by the Hamiltonian. For the case $k = 2$, this corresponds to Poinsot's construction [14]. From this perspective, the models that do not conserve the particle number, equation (9), are interesting. In terms of Poinsot's construction, the trajectories of the system are determined by the ellipsoid associated with the conserved quantity $\hat{K} = qb_1^\dagger b_1 + pb_2^\dagger b_2$ and the surface generated by the Hamiltonian. This, and the usual quantum treatment of the problem, suggest that the algebraic structure of the problem is $SU(2)$ as for the particle-number conserving case. We have been unable to explicitly construct the generators of the algebra for this case.

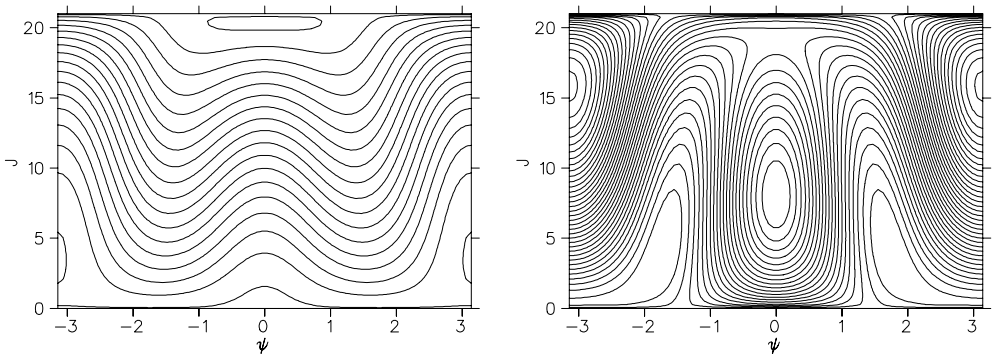

Figure 1. Two examples of the phase-space structure for distinct sets of parameters $v_{\alpha,\beta;\gamma,\delta}$ in the Hamiltonian (1). Both cases correspond to $k = 2$ and $n = 20$.

Applications of the above results are the following. First, all particle-conserving k-body interactions acting on a two-level boson system are integrable. Therefore, all particle-conserving models for two-component BEC at zero temperature are integrable. These include (for $k = 2$) the symmetric models usually considered, any non-symmetric couplings, and any perturbation (in particular with $k > 2$) that conserves the particle number. Note that the parameters of the model do influence the structure of the classical phase space (see figure 1). Secondly, the atom–molecule BEC described by Hamiltonians of the type of equation (9) [15] are integrable. However, generic perturbations may destroy the integrability, since such perturbations may not commute with the conserved quantity $\hat{K} = qb_1^\dagger b_1 + pb_2^\dagger b_2$. For example, adding a generic non-diagonal particle-conserving perturbation to equation (9) destroys integrability. Physically, this implies that the two-component and the atom–molecule BEC behave differently under generic perturbations. (Incidentally, this shows that the systems under consideration have two degrees of freedom.) Integrability implies, at the classical level, that the motion is periodic or quasi-periodic. Hence, in the semiclassical limit the evolution of a generic initial state will display oscillations in the occupation number of each single-particle level. These oscillations are nonlinear, i.e. their period depends on the energy of the initial state.

At the quantum level and fixing the polyad number $\hat{K}$, the system effectively becomes a one-degree-of-freedom system. Therefore, the spectrum corresponds to the superposition of sequences of one-dimensional nonlinear systems. Each sequence is associated with a locally stable fixed point of the classical phase space (by EBK quantization). These facts provide a semiclassical explanation of the unexpected results reported in [16], obtained in the dense limit ($n \gg l$ and k is fixed) for the k-body embedded ensembles of random matrices for bosons [16, 17]. The ensemble is defined for $k = 2$ by (1) choosing the real coefficients $v_{\alpha,\beta;\gamma,\delta}$ ($\epsilon_1 = \epsilon_2 = 0$) as the independent matrix elements of the Gaussian orthogonal ensemble [18]. Each member of the ensemble is associated with a specific classical system. Different choices of these coefficients will generically lead to topologically different structures in phase space; this accounts for the non-ergodic behaviour of the ensemble. Figure 1 illustrates the structure of the reduced phase space for fixed n and two distinct sets of coefficients. The ensemble is therefore integrable and the fluctuation properties of the spectrum of individual members of the ensemble correspond to those of a picket-fence spectrum after unfolding. The sudden localization in Fock space of certain excited states can be understood likewise.

Summarizing, we have shown that a broad class of generic two-level boson systems is integrable independently of the specific values of the parameters that define the interaction.

These systems are two-degrees-of-freedom systems. The interaction is such that it either conserves the number of particles or conserves a rational linear combination of the number of particles occupying each level. The important property that leads to integrability resides in the fact that classically these systems display at most one resonance. Integrability is destroyed generically by perturbations which allow other independent resonances to appear in phase space.

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